Wreath Products of Distributive Forest Algebras

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LICS 2018

Introduction

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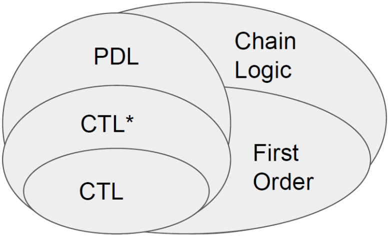
  Can a given regular language L be defined by a formula of some logic?
- in other words, to
  Give an effective characterization of the precise expressive power of the logic.
- For automata over **words**, we have effective tests for definability in many temporal and predicate logics
- For automata over **trees**, situation is quite different:
  Effectively deciding expressibility in CTL, CTL* , $FO[\prec]$, PDL, ... remains open.

Logics on Finite Trees



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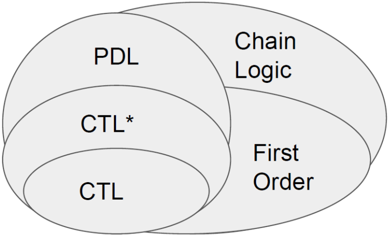

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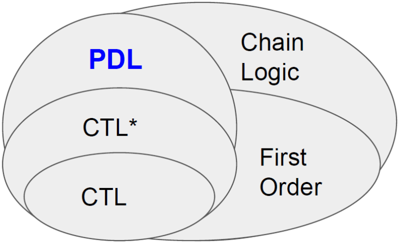

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Propositional Dynamic Logic

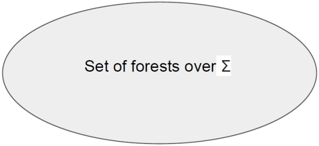
1. For any regular word language L, the language of

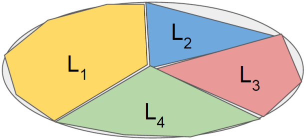
   'Forests where at least one path is in L'

   is in PDL.

Propositional Dynamic Logic

1. Let $L_1, \ldots, L_n$ PDL languages that partition the set of forests.





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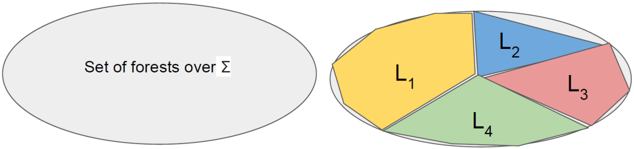

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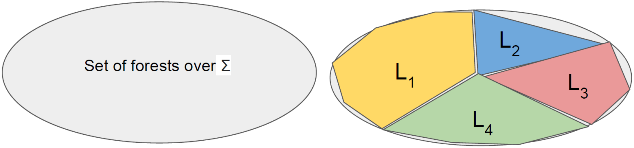
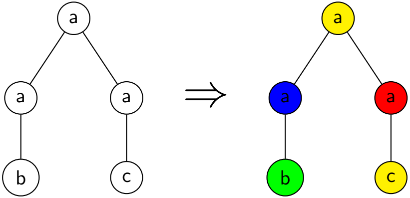


Let L a regular word language over $\Sigma \times \{a_1, \ldots, a_n\}$.
Then the following language $\mathcal{L}$ is in PDL:

Given a forest f, add to each node a label L_i if the forest below it is in L_i.

Set $f \in \mathcal{L}$ iff the resulting forest has a path in L.



(follows Bojańczyk, *et. al.*, (2012))

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Open Question

Given a regular forest language, can we decide whether it is definable in PDL?

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Following Bojańczyk, *et. al.*, (2012) and Straubing (2013), we attack this problem using algebraic characterizations in terms of **wreath products** of **forest algebras**.

Background: Forest Algebras

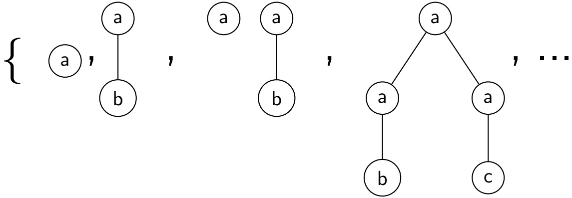
Definition (Bojańczyk and Walukiewicz, 2008)

A tuple (H, V) is called a *forest algebra* if:

1. H and V are monoids
2. There is a faithful action $V \times H \to H$
3. There is a map $I_{\cdot} : H \to V$ such that $I_h h' = h +_H h'$.
4. For each $h \in H$, there is $v \in V$ such that $h = v \cdot_V 0_H$.

Example (Free Forest Algebra)

- H = set of finite forests over some finite alphabet Σ



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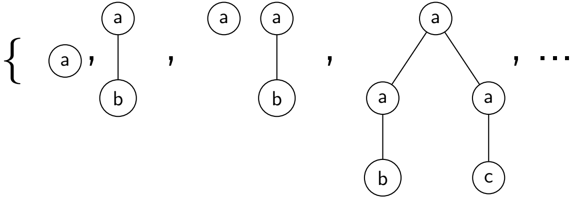
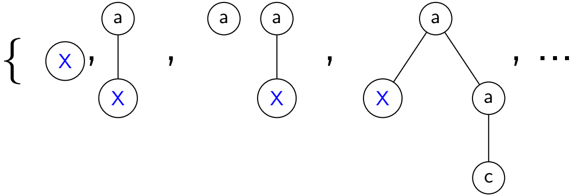


- V = set of such forests where one leaf is a variable



Example (Free Forest Algebra)

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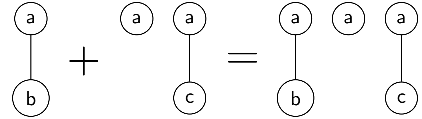
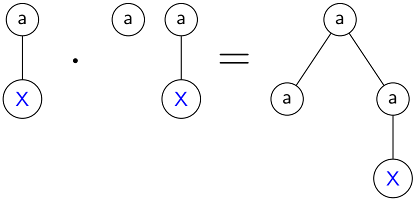
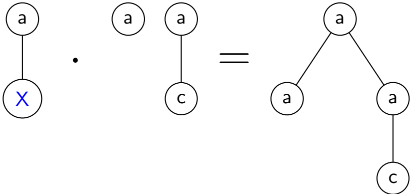


V is a monoid:



Action:



Forest Algebras and Languages

- A *forest language* is a set of forests – that is, a subset of H_Σ.
- A forest language $L \subset H_\Sigma$ is *recognized* by (H, V) iff there is a morphism

$$\phi : (H_\Sigma, V_\Sigma) \to (H, V)$$

  such that

$$L = \phi^{-1}(\phi(L))$$

Proposition

Regular forest languages are exactly those recognized by finite forest algebras.

Wreath Product

Definition (Bojańczyk, *et. al.*, (2012))

Let (H_1, V_1), (H_2, V_2) be forest algebras.

$$(H_1, V_1) \wr (H_2, V_2) := (H_1 \times H_2, V_1^{H_2} \times V_2)$$

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Multiplication:

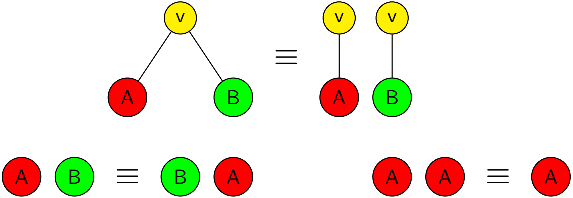
$$(f, v)(f', v') := (f'', vv')$$

with $f''(h) := (f(v'h)) \cdot (f'(h))$.

Distributive Forest Algebras

Definition (Bojańczyk, *et. al.*, (2012))

A forest algebra (H, V) is called *distributive* if any morphism into it respects the identities



- Distributive forest algebras can only distinguish forests with different *sets of paths*

Characterisation of PDL

Theorem (Bojańczyk, *et. al.*, (2012))

For a regular forest language $\mathcal{L} \subset H_\Sigma$*, the following are equivalent:*

1. $\mathcal{L}$ *is definable in PDL*
2. *There are finite distributive forest algebras* $(H_1, V_1), ..., (H_n, V_n)$ *such that* $(H_1, V_1) \wr ... \wr (H_n, V_n)$ *recognizes* $\mathcal{L}$

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Similar characterizations for other classes:

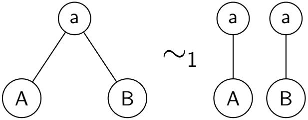
- *CTL*: only one specific distributive algebra is allowed
- *CTL**: only aperiodic distributive algebras are allowed
- ...

Iterated Distributive Laws

Definition (cf. Straubing (2013))

For each $k \geq 1$, define a congruence $\sim_k$ on $\Sigma^\Delta = (H_\Sigma, V_\Sigma)$:

1. Base case: $\sim_1$ is the congruence generated by:



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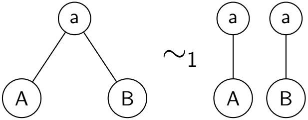

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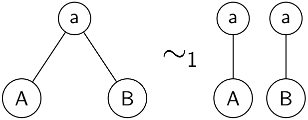

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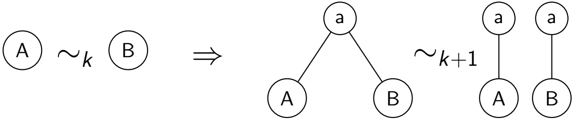

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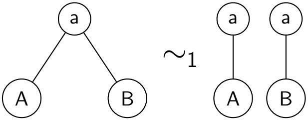

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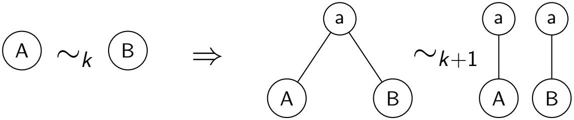

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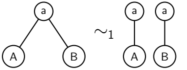

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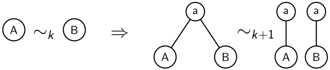

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A converse to this statement would yield a characterization of PDL!

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- Any PDL language is recognized by a k-distributive forest algebra, for some k.

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An affirmative answer would settle decidability of PDL!

Our Main Result

We solve this for $k = 2$:

Theorem

1. *Let (H, V) be finite and 2-distributive. Then every language recognized by (H, V) is definable in PDL.*
   Further, a product of 4 distributive algebras is enough.
2. *It is decidable whether a regular language is recognized by some 2-distributive forest algebra.*

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- Given a 2-distributive forest algebra (H, V),
- we seek distributive algebras $(H_1, V_1), \ldots, (H_4, V_4)$ such that

$$(H, V) \prec (H_1, V_1) \wr (H_2, V_2) \wr (H_3, V_3) \wr (H_4, V_4)$$

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We solve two sub-problems related to the left and right factors:

1. To obtain a right factor, we study the problem of separating forest languages by looking at paths only
2. We then compute a 'minimal' left factor and show that it is distributive.

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- Approach is parallel to much work on logic over words

Separation Lemma

- For a forest f, let $\pi(f)$ be the set of paths in the forest.

Lemma (Separation Lemma)

Let $\mathcal{L}_1, \mathcal{L}_2 \subseteq H_\Sigma$ be regular forest languages such that

$$\pi(\mathcal{L}_1) \cap \pi(\mathcal{L}_2) = \emptyset$$

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Further, X can be written as the wreath product of *three* distributive algebras.

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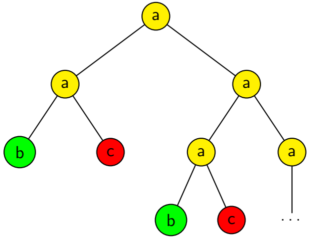
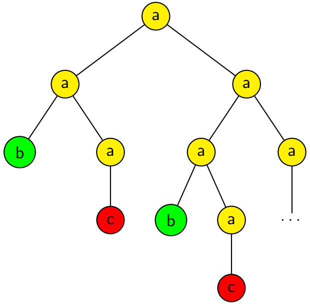
- Similar results are often used to prove decidability for logics over **words**

Separation Lemma: Example

$\mathcal{L}_1$: ‘Each path ends in siblings labeled b and c’



$\mathcal{L}_2$: ‘Each path ends in siblings labeled b and a[c]’



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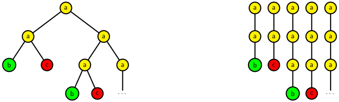

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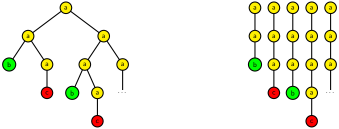

- Elements of $\mathcal{L}_1, \mathcal{L}_2$ can be told apart just from looking at the sets of paths.
- No *finite* distributive forest algebra can separate them.
- Nonetheless, PDL separates them.

Proof of the Main Result

1. Use separation lemma to construct right factor
2. Remaining problem is then to find a distributive left-hand factor (H_1, V_1) such that

$$(H, V) \prec (H_1, V_1) \wr (H_2, V_2) \wr (H_3, V_3) \wr (H_4, V_4) \qquad (1)$$

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3. Construct the 'minimal' left-hand factor (H_1, V_1)
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   - For monoids, the analog is a category (Tilson, 1987)
   - For forest algebras, result is *forest category* (Straubing, 2018)

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4. Can show that this factor can be replaced by a distributive forest algebra

Decidability of 2-Distributivity

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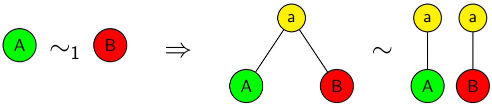
It is decidable whether a regular language is recognized by some 2-distributive forest algebra.



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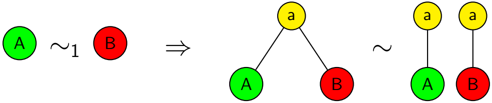
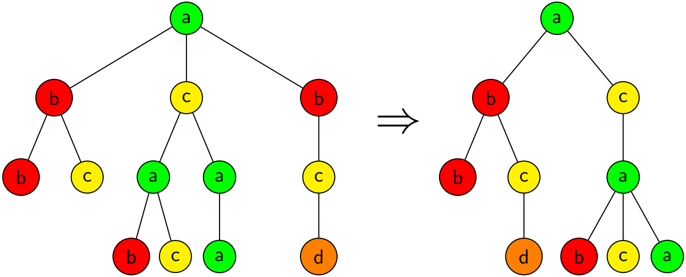


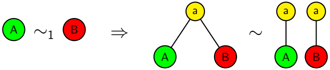
1. Transform regular forest languages into a normal form by mapping forests to representatives that only depends on the set of paths



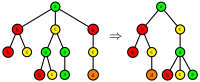
Decidability of 2-Distributivity

Theorem

It is decidable whether a regular language is recognized by some 2-distributive forest algebra.



1. Transform regular forest languages into a normal form



2. This transformation preserves regularity of languages
3. Find pairs A, B by checking intersection of images under this transformation

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- Generalizing this to $k > 2$ would settle decidability of PDL
- Builds on algebraic characterization that is similar to related logics for which decidability is open
  - Our results may shed light on this larger family of open problems
- Our proof method relates to results from the study of word languages
  - Shows how classical theory generalizes to trees and forests